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The Holistic and Partial Quantitative Reasoning Process of Students in Linking Covariation Quantities

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Abstrak: Quantitative reasoning becomes the basis for students in solving mathematical problems, especially problems related to quantities on a graph and functions. The problem is closely related to covariation. Therefore, the purpose of this research was to categorize and describe the students' quantitative reasoning processes in solving covariation mathematical problems. The process of quantitative reasoning was reviewed from APOS Theory. Research subjects were 38 high school students from three schools. The research was conducted in three stages. The first stage, students completed the Covariation Problem Solving Task while they were doing think aloud. In the second stage, the researcher evaluated the answer sheet and played back the think aloud record, then noted several important things that needed to be reconfirmed to the subject. On the third stage, task-based interview was conducted to clarify and explore data that had not been obtained through think aloud. Data were analyzed by the stages of transcribing data, reducing and categorizing data, drawing the structure of reasoning, and drawing conclusions. This study found and described that there were two categories of the covariation quantitative reasoning processes, namely holistic and partial covariation quantitative reasoning. The holistic covariation quantitative reasoning process was carried out by students through linking quantities, by coordinating the value of quantities from the whole part. On the other hand, the partial covariation quantitative reasoning process was applied by partitioning the whole part to determine the quantity value, then coordinating the quantities based on the quantities of the part.

Keywords: APOS Theory; Covariation Problems; Mathematical Problem Solving; Quantitative Reasoning.

Abstrak: Penalaran kuantitatif menjadi dasar bagi siswa dalam menyelesaikan masalah matematika, terutama masalah yang berkaitan dengan besaran pada grafik dan fungsi. Masalah ini sangat berkaitan dengan kovariasi. Oleh karena itu, tujuan penelitian ini adalah untuk mengkategorikan dan mendeskripsikan proses penalaran kuantitatif siswa dalam menyelesaikan masalah matematika kovariasi. Proses penalaran kuantitatif ditinjau dari Teori APOS. Subjek penelitian adalah 38 siswa SMA dari tiga sekolah. Penelitian dilakukan dalam tiga tahap. Tahap pertama, siswa menyelesaikan Tugas Penyelesaian Masalah Kovariasi sambil melakukan think aloud. Pada tahap kedua, peneliti mengevaluasi lembar jawaban dan memutar kembali rekaman think aloud, kemudian mencatat beberapa hal penting yang perlu dikonfirmasi kembali kepada subjek. Pada tahap ketiga, dilakukan wawancara berbasis tugas untuk mengklarifikasi dan mengeksplorasi data yang belum diperoleh melalui think aloud. Data dianalisis melalui tahapan transkripsi data, reduksi dan kategorisasi data, penyusunan struktur penalaran, dan pengambilan kesimpulan. Studi ini menemukan dan menjelaskan bahwa terdapat dua kategori proses penalaran kuantitatif kovariasi, yaitu penalaran kuantitatif kovariasi holistik dan parsial. Proses penalaran kuantitatif kovariasi holistik dilakukan oleh siswa melalui pengaitan besaran, dengan mengkoordinasikan nilai besaran dari bagian keseluruhan. Di sisi lain, proses penalaran kuantitatif kovariasi parsial diterapkan dengan membagi bagian keseluruhan untuk menentukan nilai besaran, kemudian mengkoordinasikan besaran berdasarkan besaran bagian tersebut.

Kata kunci: Teori APOS; Masalah Kovariasi; Penyelesaian Masalah Matematika; Penalaran Kuantitatif.

INTRODUCTION

Mathematics learning in Indonesia has rules and standards that serve as a reference in the implementation process and expected goals. The Indonesian Education Ministerial Regulation number 22 of 2006 concerning content standards (Permendiknas No. 22, 2006) said that reasoning is one of the standards in mathematics learning. This is also stated in NCTM (2000) which reveals reasoning as a standard in mathematics learning. Reasoning in this case is defined as a thought process that is logical and analytic in solving problems in the process of drawing conclusions. In mathematics, one object of reasoning is about quantities relationship (NCTM, 2000). In this case, the quantities' relationship is built into quantitative reasoning that involves quantitative operations in solving problems to obtain a solution (Thompson, 1993).

The development of quantitative reasoning ability is not easy and is not fast done (Smith & Thompson, 2007). Students who are capable of producing completion from various experiences over the years will be more successful in understanding algebra because they makes arithmetic and algebraic knowledge more meaningful and productive. Koedinger & Nathan (2004) state that if students involve quantitative reasoning in representing a problem, then there is a positive change in the problem representation of cognitive processes and performance aspects in algebra. For example, students represent quantitative reasoning in various forms including algebraic and arithmetic symbols, equations, images, and verbal (Swastika et al., 2020).

Previous quantitative reasoning research tends to be associated with other reasoning, that is consistent with the problem (task) used at the time of the study. Some types of reasoning include algebraic reasoning (Caglayan, 2013; Hackenberg & Lee, 2015; Ikram et al., 2020), proportional reasoning (Hilton et al., 2016; Lobato & Siebert, 2002), transformational reasoning (Johnson, 2012), and covariational reasoning (Carlson et al., 2002; Carlson et al., 2015; Ellis, 2007; Johnson, 2015; Stalvey & Vidakovic, 2015; Syarifuddin et al., 2019a, 2019b, 2020; Weber et al., 2014; Wilkie, 2020; Yemen-Karpuzcu et al., 2017). Therefore, in this study, the problem studied was the problem of covariation. Covariation problem contains algebraic problems (arithmetic and algebraic operations) and problems involving ratios or comparisons.

Then several studies of covariation problems in high school students emphasize the notion of a covariation quantities correlation from one subject, which mean no description of other subjects is revealed (Johnson, 2012). In addition, other studies only proves the comparison of the equations they produce to be generalizable to the values of quantities with the same comparison and they emphasize the importance of learning about quantitative reasoning in influencing student generalizations (Ellis, 2007). To refer to some of the research studies, this study examined the different processes of high school students in forming a quantities' relationship in producing a Cartesian chart, as well as interpreting the shape of the quantities' relationship resulted from the graph. The subject selection of high school students was due to the difficulties experienced by students studying calculus, so that this study students could employ high school students as subjects to prepare students for dealing with calculus learning.

The students' quantitative reasoning process in solving variate problems was investigated using APOS theory (Action, Process, Object, and Schema). APOS was used as an assessment tool to investigate how mental construction arose from students' reasoning on a problem (Burns-Childers & Vidakovic, 2018). Assumptions on individual mathematical knowledge were reflected on the students' responds to situations of perceived mathematical problems and their solutions in social contexts and their mental construction structures in certain situations.

LITERATURE REVIEW

Quantitative Reasoning

Quantitative reasoning involves analyzing quantities and relationships among quantities in a situation, creating new quantities, and making conclusions of quantities properties (Ramful, 2009; Thompson, 1993). Then according to Moore (2014), quantitative reasoning is a student's mental action to understand the situation, build a situation image of quantity, and the reasoning about the relationship among quantities.

Quantitative reasoning requires quantitative mental operations, such as in forming multiplicative comparisons of addition comparisons and combination (Thompson, 1995). Quantitative reasoning also requires relational reasoning about quantitative structures, which requires rules of mental operation to understand situational quantities and rules of operation that mentally allow one to recognize quantity as one whose value varies. Quantitative reasoning can be numerical or non-numeric because one does not really need to know the size of the quantities or actually measure the quantities for quantitative reasons (Thompson, 1993). While Thompson (1994) explains that quantitative (non-numerical) operations create quantities, whereas numerical operations are used to evaluate quantities. This opinion is made clear by (Lobato & Siebert, 2002) that quantitative operations are non-numeric, while numerical ones (arithmetic operations) are used to evaluate quantities.

Quantitative reasoning refers to ways of thinking that emphasize a student's cognitive development of conceptual objects with which students reason about certain mathematical situations (Smith & Thompson, 2007). On the other hand, NCTM (2000) provides an overview of the objectives of quantitative reasoning, namely developing the ability to analyze quantitative information and determining skills and procedures that can be applied to a particular problem to a solution. Therefore, it is not limited to the skills acquired in mathematics, but it also includes reasoning abilities that are developed over time through practice in almost all school or college programs, as well as in daily activities.

Mathematical Problem Solving

Student success in solving problems is an important goal in learning mathematics. Problem solving strategies are actions or methods used by students to understand and solve a problema (Basri et al., 2019; Ikram et al., 2020; Sriraman, 2003). There are even several stages that students must pass so that students are successful in solving problems. Carlson & Bloom (2005) explain that there are four stages of multidimensional problem solving framework, namely: orientation, planning, implementation, and checking. While the stages of problem solving according to Polya (1985) consist of (1) understanding the problem; (2) developing a plan; (3) implementing the plan; and (4) looking back (reflection).

Quantitative reasoning leads to problem solving (Dwyer et al., 2003). In this case, Dwyer et al. say that the problem solving components are understanding and defining the problem, solving the problem, and understanding the results. In the process of solving problems, quantitative reasoning can use quantitative operations (Thompson, 1994), in which quantitative operations (Lobato & Siebert, 2002) are operations that are non-numerical, while those that are numerical (arithmetic operations) are used to evaluate quantities. The stages of covariation problem solving in the process of quantitative reasoning (Dwyer et al., 2003) occur through six steps namely understanding information in a given task, choosing mathematical methods for problem solving, performing problem solving, inferring and communicating quantitative information, interpreting information related to quantities, as well as checking and estimating answers.

APOS Theory in Solving Covariation Problems

APOS (Arnon et al., 2014) is an acronym from the abbreviation of a type of mental structure consisting of Action, Process, Object, and Schema. APOS theory is a theoretical reference frame of mathematics education that cognitively describes how students build or learn concepts (topics) in mathematics based on their previous mathematical structure, which in turn evolves to form other knowledge (García-Martínez & Parraguez, 2017). A similar statement was delivered by Arnon et al. (2014) that learning mathematics must be based on helping students to use the mental structures they already have and developing new structures that are stronger to handle more and more advanced mathematics.

The flow structure and mental mechanisms follow a spiral approach (Dubinsky, 1997). This means that after the individual constructs the object as a result of encapsulation, then the object can be reassigned to form actions at a higher level. The action is interiored into a process and then the process is encapsulated into objects. In APOS Theory, schemes owned by individuals constitute a collection of actions, processes, objects, and other schemes (Arnon et al., 2014). Schemes that have been built can be stimulated again into objects through mental mechanisms of thematization.

In this study, APOS theory is a framework consisting of mental structures (actions, processes, objects, and schemes) and mental mechanisms (interiorization, coordination, reversal, encapsulation, de-encapsulation, and thematization) that are used to investigate mental construction appearing in students' quantitative reasoning on the covariation problem.

METHOD

This study described and categorized the quantitative reasoning of students in solving mathematical problems related to covariation problems. The process of building quantitative reasoning was seen from the way students completed CPST, with a descriptive exploratory approach.

Participants

The subjects in this study were high school students in Indonesia from different regions, namely in the Bima district, Bima town, and Malang city. The study was conducted to the students in grades 11 and 12 who had studied material related to the assignment used for research data collection. From these three schools, 38 subjects were obtained that produced a quantitative reasoning process on the covariation problem. The results distribution of the research subjects selection for each category is presented in Table 1.

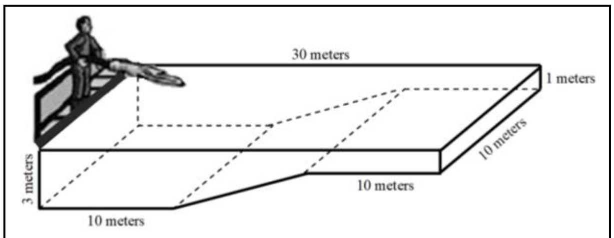
Table 1. Selection Results Distribution of Research Subjects

No	School	The number of subject candidate on each category	
		Holistic quantitative reasoning	**Partial quantitative reasoning**
1	A senior high school in Bima District	10 students	1 student
2	A senior high school in Bima Town	8 students	1 student
3	A senior high school in Malang City	15 students	3 students
	The number	33 students	5 students

Based on a hypothetical theoretical framework, data were grouped based on holistic and partial quantitative reasoning. Each onne subject represented holistic and partial covariation quantitative reasoning process presented and analyzed was taken. Subject selection was carried out to the level of data saturation. Data saturation in this case was obtained by paying attention to each subject from each category having the same pattern of several research subjects.

Instrument and Procedures

Research data were collected from several sources, including supporting and main instruments. The supporting instruments in the form of CPST were shown in Figure 1, while the main instrument was the researcher (Creswell, 2012). The quantitative reasoning process of CPST in terms of the APOS stages carried out by the subjects is based on the indicators in Table 2.



Someone fills the water in swimming pool with the water debit that is out of hose is 6 cubic meters of water in 3 minutes, as shown in the figure. When the pool is full of water, what is the graph shape of volume and time? What can you conclude from the relationship of volume and time?

Figure 1. Covariation Problem Solving Task (CPST)

The process in this study began with the selection of research subjects. In this subject selection process, researchers explained and trained think aloud to prospective subjects, so they could do think aloud well when the completed CPST. When practicing think aloud, prospective subjects were given easy questions, namely determining the area of a block that was known to be 8 cm long, 3 cm wide, and 2 cm high. Furthermore, prospective subjects answered these questions while they did think aloud. Prospective subjects completed CPST while they did think aloud with the aim to find out the student's quantitative reasoning process. When the prospective subjects were working, the researcher observed them from a distance (outside the room). Task-based interview was conducted if there was information that had not been obtained or information that needed to be confirmed based on the results of think aloud with the aim to categorize prospective subjects into predetermined research indication categories, namely quantitative reasoning, holistic covariation and partial quantitative reasoning. Data saturation was data from subjects of each category having the same or fixed patterns from several research subjects taken.

Table 2. Indicators of the stages of quantitative reasoning in covariation problem solving based on APOS Theory

Mental Structure	Quantitative Reasoning Stage	Indicator	Mental Mechanism
Action	Understanding Information	• Undestanding and Writing the sides of the pool.	Interiorization
		• Identifying and writing the quantities involved, those are length, width, height, volume, and time.	Interiorization
		• Writing new quantities from the pool partition, namely yaitu length as *l*, width as *w*, height as *h*, volume as *V*, dan time as *T* (*t*).	Interiorization
		• Writing the size of quantities from the pool whole sides.	Interiorization
Process	Choosing mathematics method to solve problems.	• Partitioning and mentioning the shapes of pool, such as there is when the pool shaped cube or trapezoidal prism.	Coordination
		• Writing quantities size of pool partitioned side.	Coordination
		• Determining pool volume formula.	Reversal
		• Determining the volume of partitioned geometry formula	Reversal
		• Writing the steps/formula of determining the time spent for fulfilling the whole part of the pool.	Reversal
	Problem solving process.	• Determining the volumeof each geometry.	Coordination
		• Determining water debit flowing in the hose and time spent to flow the water.	Coordination
		• Searching and counting the volume of pool sides by substituting the quantities' values into appropriate formula.	Coordination
		• Determining the time spent for fulfilling the pool with water.	Coordination
		• Adding the volume of partial pool geometry to obtain the whole pool volume.	Coordination
		• Determining coordinate points to visualize the	Coordination

		graph.	
Objek	Concluding and communicating quantitative information	• Visualizing the graph describing the relationship between time and volume quantity of water.	Encapsulation
	Interpreting information.	• Describing the relationship role of time volume quantities from the resulted graph.	Encapsulation
	Checking and predicting the answers	• Rechecking carefully the coordinate point and determining the interval of the quantities in the graph.	Encapsulation
		• Reobserving the description of volume and time relationship resulted from the graph or the process.	Encapsulation
Schema	Understanding the concept of quantitative information.	• Understanding and mentioning/writing the mathematical concept involved in covariation.	Thematization

Data Analysis

The data analyzed were data obtained from subject answer sheets, think aloud records and interviews, and field notes throughout the research process. Data were analyzed by combining several data sources with retrospective analysis. In distinguishing the reasoning process produced by the subject, a comparative analysis (Corbin & Strauss, 2012) was carried out. In the data analysis stage, activities undertaken after obtaining data included: transcribing think aloud and interview data, reducing data, encoding data, describing students' thinking processes in generating quantitative reasoning in covariation problem solving, and drawing conclusions (Creswell, 2012). The data encoding in the study are as shown in Table 3.

Table 3. Encoding for Quantitative Reasoning Stages in Covariation Problem Solving based on APOS Theory

Code	Term	Code	Term
	Initial and final activity		Process
	Understanding information		Object
	Choosing mathematics method for solving problem		Scheme
	Problem Solving Process	Int	Interiorization
	Concluding and communication quantitative information	Coord	Coordination
	Interpretating information	Rev	Reversal
	Checking and predicting the answer	Encap	Encapsulation
	Problem solving activity order	De-Encap	De-Encapsulation
	Action	Tem	Thematization

RESULTS

This section presents and analyzes data collected from the research process. The participating subjects were high school students in Bima district, Bima town, and Malang city. From the three schools, as many as 38 prospective subjects worked on CPST with think aloud. After evaluating the answer sheets and re-observing the results of the think aloud, 33 subjects showed the covariation quantitative reasoning process by paying attention to the quantities relationship as a whole, and 5 subjects showed the covariation quantitative reasoning process by observing the relationship of the quantities.

To recognize the covariation quantitative reasoning process and the stages occurring, several samples representing the characteristics for task-based interviews were choosen. From the answer sheets, it was obtained that the results of think aloud and in-depth interviews showed 2 categories of covariation quantitative reasoning, namely: 1) the process of holistic covariation quantitative reasoning, and 2) the process of partial covariation quantitative reasoning.

The subjects presented were choosen to represent the same characteristics from each category. The holistic covariation quantitative reasoning process was explained by one subject, namely *Ada* (initial name), then for the partial covariation quantitative reasoning process was also presented by one subject named *Aysh* (initial name). Data presented and analyzed were obtained from the subject's answer sheets in completing CPST, think aloud, task-based interviews, and observation note. Interview in this case was conducted by researcher (R) of the subject.

Holistic Covariation Quantitative Reasoning Process

This section describes and analyzes data from *Ada*, which describes the structure of reasoning in solving CPST problem based on the stages of quantitative reasoning that were coupled with analysis using APOS Theory, which included mental structures and mental mechanisms. The stages of quantitative reasoning began with reading and understanding information, then determining mathematical methods to solve mathematical problems, solving problems with predetermined methods, concluding and communicating quantitative information, interpreting information, estimating and examining the feasibility of answers, and understanding concepts from quantitative information.

The initial mental structure carried out by *Ada* was the stage of action, in which *Ada* did the interiorization by reading and understanding information through paying attention to the problem of the problem editor, pictures, and commands on the task. In think aloud, *Ada* started completing the assignments by reading the instructions. In this case, *Ada* had identified the quantities existing in the pool. There was then a problem solving plan by paying attention to the shape of the pool to determine the mathematical method for solving the right problem.

Next, there was initiating problem solving planning on the mental structure of the process, namely by conducting mental coordination mechanisms through paying attention to the shape of the pool to determine mathematical methods for solving appropriate problems. To determine the relationship of volume quantity and time in graphical form, *Ada* planned to first calculate the pool volume by dividing the pool into several shapes. The following is the excerpt statement from think aloud data.

Ada : *So, my steps to solve this task is, initially by calculating the pool volume first through dividing the geometry into three parts* [while pointing the pool shape in the task]. *In order to ease the calculation and the division, I am going to draw it first* [Subject drew the pool shape and divided it into three parts], *here I divided it into three geometries, the fisrt, the second and the third*.

The think aloud results were reinforced by the work of Ada on the answer sheet in mentioning the division of the pool into three different spatial structures, as shown in Figure 2.

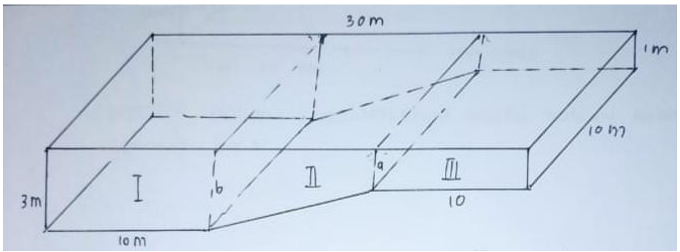


Figure 2. *Ada's* work results dividing the pool and identifying the geometries of the pool

In this mental coordination mechanism, *Ada* mentioned the shape of geometry was partitioned or divided from the pool. It was stated that the shape of the first geometry is a beam-shaped geometry, then the second geometry was in the form of a trapezoidal prism, while the third geometry was a beam-shaped one. After doing the mental coordination mechanism, then *Ada* continued the mental reversal mechanism by recalling mathematical problem solving methods that were suitable in solving problems. *Ada* mentioned that in order to get the overall pool volume, the subject should calculate the volume of each geometry. To calculate the volume of geometry I (beams), *Ada* used the formula $V = l \times w \times h$, the volume of geometry II (trapezoidal prism) was obtained by using the formula $V = [((a + b) / 2) \times H)] \times h$, and the volume of geometry III was obtained from the formula as the first geometry, $V = l \times w \times h$.

After determining the mathematical method to solve the problem or determining the formula through a mental reversal mechanism, then you would go back to the coordination process. In this case, *Ada* had to substitute the quantities value into the formula corresponding to each geometry. The volume of geometry I was obtained with V = 10 m × 3 m × 10 m = 300 m^3, then the volume of geometry II was obtained with $V = [((1 + 3) / 2) \times 10)] \times 10 = 200$ m^3, and the volume of geometry III was obtained with V = 10 m × 1 m × 10 m) = 100 m^3.

This was reinforced by the problem solving done by *Ada* on the answer sheet in stating the shape of the geometry and the process of problem solving, as shown in Figure 3.

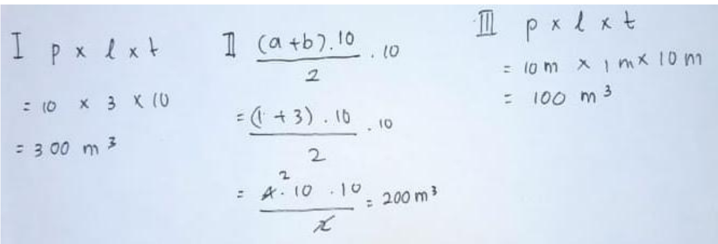


Figure 3. The *Ada's* work in the process of covariation problem solving

After obtaining the volume for each geometry, *Ada* continued the mental coordination mechanism by calculating the overall pool volume by adding up the volume of the geometry I, geometry II, and geometry III. The volume of the pool is V_{pool} = 300 m^3 + 200 m^3 + 100 m^3 = 600 m^3. This can be seen from the think aloud transcript by *Ada* below.

Ada : The volume total of the pool is obtained from the addition of every geometry volume, which is 300 added with 200, then added with 100, and the total is obtained in the amount of 600 m^3.

This can also be seen from the results of *Ada's* work in adding up the pool volume from different geometry, as shown in Figure 4.

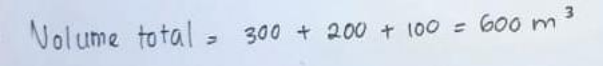


Figure 4. The result of *Ada's* work adding up the volume of the geometries of the pool

The next problem solving done by *Ada* was re-establishing the mental coordination mechanism to determine the time needed to fill the volume until it was fully charged. The method used by *Ada* was to look for water discharge released from the hose for 1 minute, so that 2 m^3 per minute (2 m^3/minute) was obtained. *Ada* divided the full volume of the pool by the discharge of water that came out per minute, and obtained time in the amount of 300 minutes. The work on the answer sheet in determining the time to fill the pool until it was fulfilled, as shown in Figure 5.

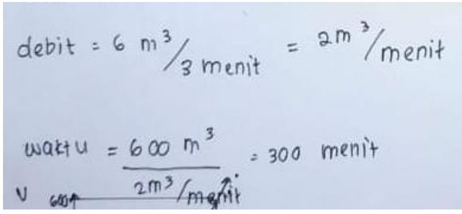


Figure 5. The result of *Ada's* work in determining the time to fill the pool to the fullest

Determination of the pool volume and the time needed to be fully charged had been determined by *Ada*, so that the next stage carried out by *Ada* was on the mental structure of the object. *Ada* encapsulated by drawing a graph of the relationship between volume and time. The time quantity was placed on the *x*-axis, while the volume quantity was placed on the *y*-axis. In drawing graph, there was a mental coordination mechanism involved in connecting coordinate points between the volume value of 600 m^3 and 300 minutes, connected by a straight line. As seen from the results of *Ada's* think aloud in the following.

Ada : *We draw a graph, a graph relating to time and volume to fill the pool* [beginning to draw the graph], *this is the volume* [while writing *V* symbol on the *y*-axis], *this is the time* [while writing *t* symbol on *x*-axis]. *Therefore, the time needed to fill 600 m^3 of the pool is in the amount of 300 minutes.*

In the Figure, *Ada* wrote the interval for each coordinate axis. On the *x*-axis, the intervals were 100, 200, and 300 and on the *y*-axis, the intervals were 100, 200, 300, 400, 500, and 600. These intervals were determined arbitrarily without connecting their coordinate points. As shown in Figure 6.

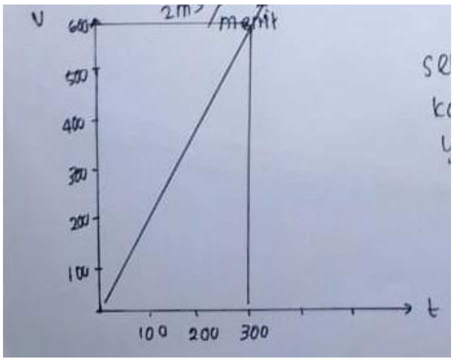


Figure 6. The results of *Ada's* work in drawing the graph showing relationship of volume and time

To find out the thought process in determining the intervals and pairs of coordinate points written on the intervals of each axis, interview was conducted. When the subject was asked for how much time the pool needs to fill the volume of 100 m^3, then *Ada* answered that it required 50 minutes and explained that 50 minutes was obtained from 100 m^3 divided by 2. In this process, *Ada* did the de-encapsulation mechanism, this can be seen from the interview transcript conducted in the following.

R : How much this quantity? [while pointing the value 100 on y axis or volumen axis].
Ada : 100 (100 m^3)
R : So how much time does it require to fulfill 600 m^3 of water? [while pointing the value 600 m^3 on y axis].
Ada : 300 minutes.
R : If the volume is 100 (100 m^3), how many minutes does it require to fulfill the pool?
Ada : 50 minutes.
R : How come?
Ada : It is the half of previous value
R : Which half do you mean?
Ada : It is the half of this 100, so on each minute the water volume in the pool is 2 m^3.

In the interview, there was a de-encapsulation to estimate and check the feasibility of the answer in determining the intervals and coordinate points by improving the graphic image and redrawing the appropriate intervals and coordinate points. This can be seen from the resulting image as in Figure 7.

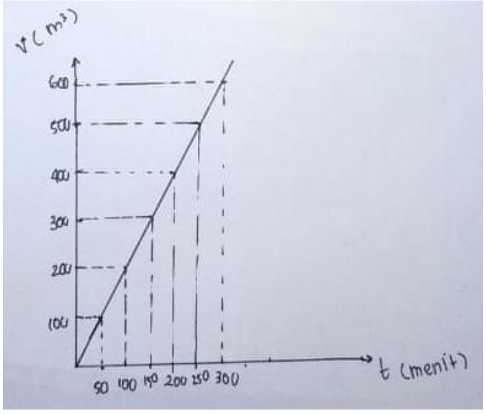
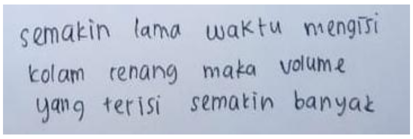


Figure 7. The result of *Ada's* work in perfecting the graph of volume and time relationship

On the mental structure of the next object, *Ada* re-did the mental structure of encapsulation to interpret the information. *Ada* described the relationship formed from the volume and time quantities, *Ada* wrote that the longer the time to fill the pool, the more the filled volume. As seen from *Ada's* work in Figure 8.



English:
The longer the time it needs to charge the pool, then the more the filled volume

Figure 8. The result of *Ada's* work concluding the relationship between volume and time quantities

Then to see the concept of *Ada's* thinking about the graph, the researcher asked the shape of the graph obtained from the relationship between volume and time as seen in the following interview transcript.

R : *What kind of graph is this?*
Ada : *Graph that goes up…*
R : *Then this is parabola graph, right?* (while creating a sample of parabola graph)
Ada : *Garis lurus, atau persamaan garis lurus*.

Thus, the mental structure of the schema formed from *Ada* was in the form of linear graph concept and its equation. In this case, *Ada* did thematization by mentioning the form of a graph in the form of a straight line and mentioning the linear equation of the graph.

Ada's process of quantitative reasoning in covariation problem solving in mental structure and mechanism holistically is presented in Figure 9.

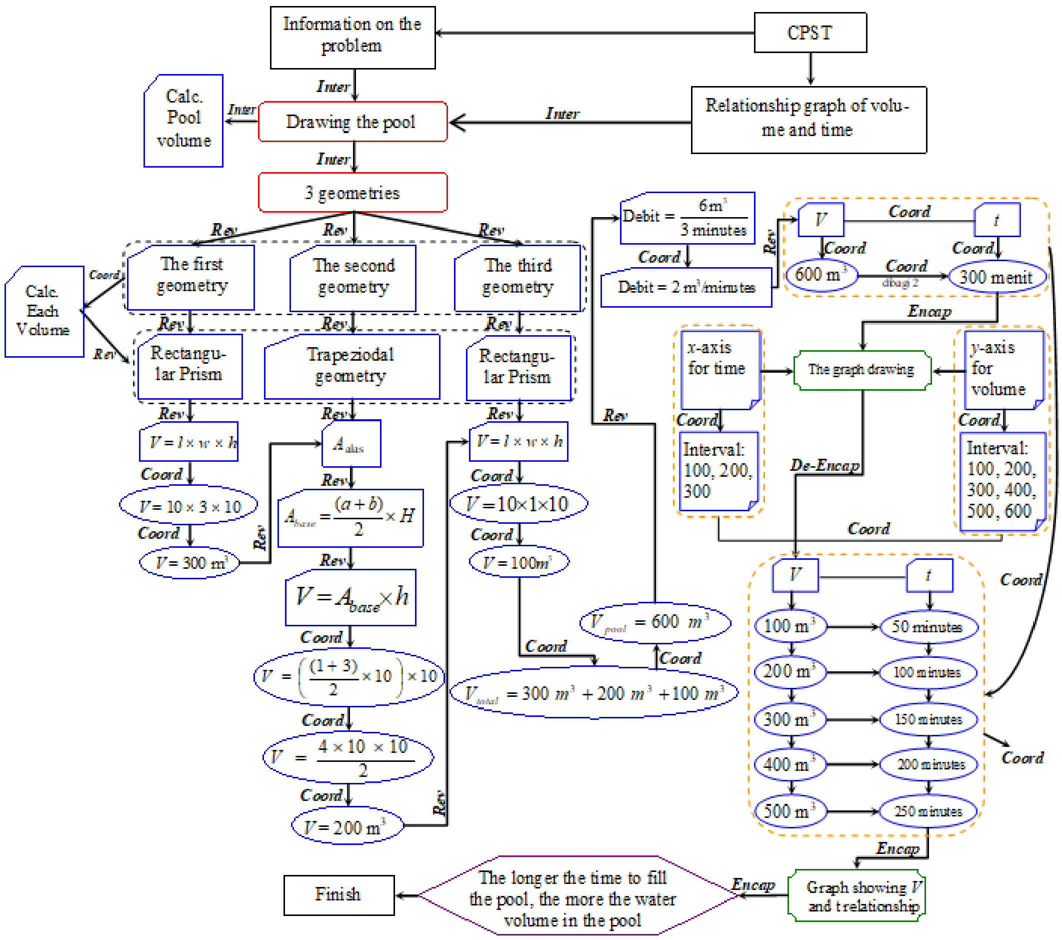


Figure 9. The Structure of Holistic Quantitative Reasoning in Covariation Problem Solving according to APOS Theory

Partial Covariation Quantitative Reasoning Process

This section describes and analyzes Aysh's data, which breaks down the structure of reasoning in solving CPST based on the stages of quantitative reasoning coupled with analysis using APOS Theory as has been done in the holistic quantitative reasoning process above. The stages carried out by *Aysh* began with the mental structure of the action. *Aysh* performed the same interiorization mental mechanism as *Ada*, which was reading and understanding information through paying attention to the problems of the task editor, images, and commands existing on the task. In think aloud, *Aysh* began to complete the task by reading the task, in this case through identifying the quantities in the pool. Then plan the problem solving strategy by paying attention to the shape of the pool to determine mathematical methods to solve the right problem.

The stage of planning the problem solving by *Aysh* was done at the mental structure stage process by conducting a coordination mechanism. In the coordination mechanism, *Aysh* planned problem solving by paying attention to the shape of the pool to determine mathematical methods to solve the right problem. To determine the relationship of volume and time quantities in graphical form, *Aysh* began to do the coordination mechanism by determining the flow of water coming out of the hose every 1 minute. The following is the statement in the excerpt of think aloud data.

Aysh : It is known that the debit of water = 6 m^3/3 minutes, meaning that the debit is 2 m^3/minute.

Think aloud was reinforced by *Aysh's* work on the answer sheet in determining the flow of water coming out of the hose, as shown in Figure 10.

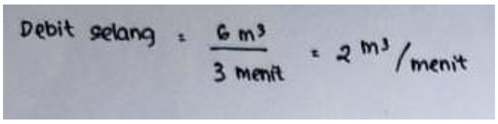


Figure 10. Aysh's work in determining the flow of water coming out of the hose

The mental structure of the next process was to carry out a coordination mechanism to find the volume of the pool through partitioning the pool as done by *Ada*. However, *Aysh* partitioned the shape of the pool by paying attention to the horizontal shape and partitioned it into two different shapes and was drawn separately for each of them. The first form was a trapezoidal prismal geometry and was referred to as geometry A and the second form was a beam-shaped geometry and referred to as geometry B. The following is *Aysh's* statement in the think aloud data excerpt.

Aysh : In the following picture (while pointing to the pool image in the task) has two geometries, we can name the bottom as A and the top part as B. It is known that the lower part is a trapezoidal prism geometry and the upper geometry is a beam geometry.

The results of the think aloud are reinforced by the work on the answer sheet in planning the problem solving as shown in Figure 11.

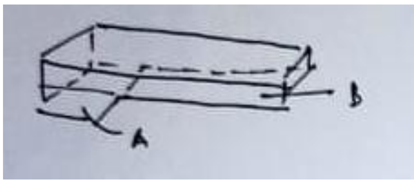


Figure 11. The results of *Aysh's* Work in planning problem solving

After partitioning the pool into two parts of geometry, *Aysh* returned to the mental mechanism of interiorization by identifying each quantity in the geometry. These quantities were mentioned together while determining the mathematical method for solving mathematical problems, in this case solving and calculating the volume of each geometry. Furthermore, *Aysh* recalled mathematical problem solving methods to solve problems. In this case, determining the formula for finding the pool volume by finding the volume of each geometry that had been separated. This can be seen from the excerpt of the think aloud statement and it was reinforced by the work in Figure 12.

Aysh : First we must look for the volume on each geomtry. The geometry A uses trapeziodal prism volume formula, whereas the geometry B utilizes beam volume.

The mental reversal mechanism was carried out by recalling the mathematical problem solving method through determining the formula for each volume of geometry, which was coupled with the coordination process by substituting the quantity value into the volume formula of geometry. *Aysh* determined the formula for geometry A volume (trapezoidal prism volume), which was $V = A_{base} \times h = ((a + b) / 2 \times T) \times h$. To determine the value of the volume, *Aysh* identified the existing quantities, in which the parallel sides a = 20 meters, b = 10 meters, and height (h) = 2 meters. The volume of the trapezoidal prism was obtained by substituting the quantities that had been identified in the formula, which was volume $V = ((20 + 10) / 2 \times 2) \times 10 = 300$ m^3. *Aysh* went on to find the volume of geometry B by determining the beam volume formula $V = l \times w \times h$ and calculating the volume by substituting the quantities value l = 30 meters, w = 10 meters, and h = 1 meter into the formula, so that it was obtained $V = 30 \times 10 \times 13 = 300$ m^3. *Aysh's* work in solving the problem of finding pool volumes as shown in Figure 12.

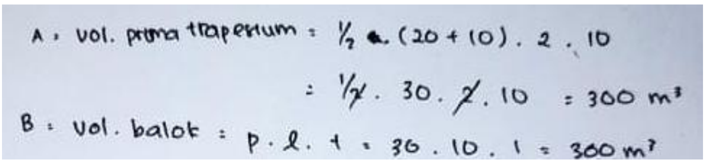
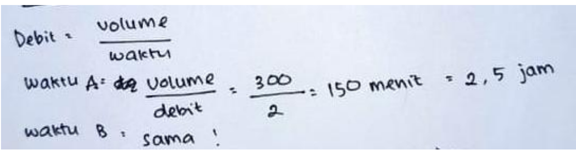


Figure 12. *Aysh's* work in finding pool volume

After obtaining the volume of geometry A and B, then *Aysh* used a reversal mechanism to recall about how to determine the time of each part. *Aysh* determined the time using the debit formula i.e. Debit $= \frac{\text{volume}}{\text{time}}$, so time $= \frac{\text{volume}}{\text{debit}}$. The debit used was the debit obtained from Figure 12. Then *Aysh* carries out a coordinating mechanism by substituting known quantities into the formula. The time for geometry A was $\frac{300\, m^3}{2\, m^3/minute}$ = 150 minutes or 2.5 hours, while the time for geometry B was $\frac{300\, m^3}{2\, m^3/minute}$ = 150 minutes or 2.5 hours. This can be seen from *Aysh's* think aloud in the following transcript.

Aysh : To determine the time, we can use the debit formula, the formula is volume quantity divided with time quantity. The time spent to fill geometry A is 300/2 = 150 minutes, to be converted into the hour unit is 60 × 2 + 30 meaning two and a half hours (2.5 hours). Time for geometry B is the same time because the volume is the same [same as geometry A].

Think aloud carried out by *Aysh* was reinforced by *Aysh's* work in determining the time as shown in Figure 13.



English:
Debit $= \frac{\text{volume}}{\text{time}}$
Time A = $\frac{\text{volume}}{\text{Debit}} = \frac{100}{2}$ 150 *minutes* = 2.5 hours
Time B = Time A

Figure 13. *Aysh's* work in determining time

The next stage was the stage in the mental structure of the object, in which after obtaining the value of volume and time quantities for each geometry A and geometry B, *Aysh* performed a mental encapsulation mechanism to deduce and communicate quantitative information by drawing a graph of the coordinate points. The coordinate point of geometry A was (2.5 hours, 300 m^3) and the coordinate point of geometry B was (5 hours, 600 m^3), the value was passed from the coordinates of geometry A. This can be seen from the interview excerpts conducted by researcher with *Aysh* below.

R : How do you determine the coordinates of this graph?

Aysh : First I determine the time in this section [pointing to the *x*-axis] and the volume in this section [pointing to the *y*-axis]. Geometry A has time quantity of 2.5 hours and volume quantity of 300 m^3, and geometry B also takes 2.5 hours and added with the existing geometry A to be 5 hours. The value was also added 300 m^3 of geometry A, to be 600 m^3.

The interview excerpt was strengthened by *Aysh's* work in drawing graph as shown in Figure 14.

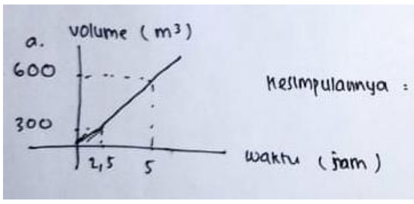
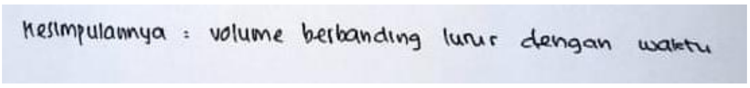


Figure 14. *Aysh's* work on drawing the graph revealing the relationship of volume and time

In the next step, *Aysh* implemented an encapsulation mechanism to interpret, estimate, and check the feasibility of the answers. *Aysh* summed up the pool volumes of geometry A and geometry B to determine the volume of pool filled with water and considered the time needed, then *Aysh* compared with the graph that had been generated. From the graph, *Aysh* concluded that volume quantity was directly proportional to time quantity. This can be seen from the result of *Aysh's* work in Figure 15 in explaining the relationship between volume and time.



English:
The conclusion: Volume quantity has positive correlation with time quantity

Figure 15. *Aysh's* work concludes the relationship between volume and time

Thus, the mental structure of the scheme formed from *Aysh* was in the form of a linear graph concept. In this case, *Aysh* conducted thematization by stating that the volume was proportional to time.

Aysh's quantitative reasoning process in covariation problem solving in mental structures and mental mechanisms are partially presented in Figure 16.

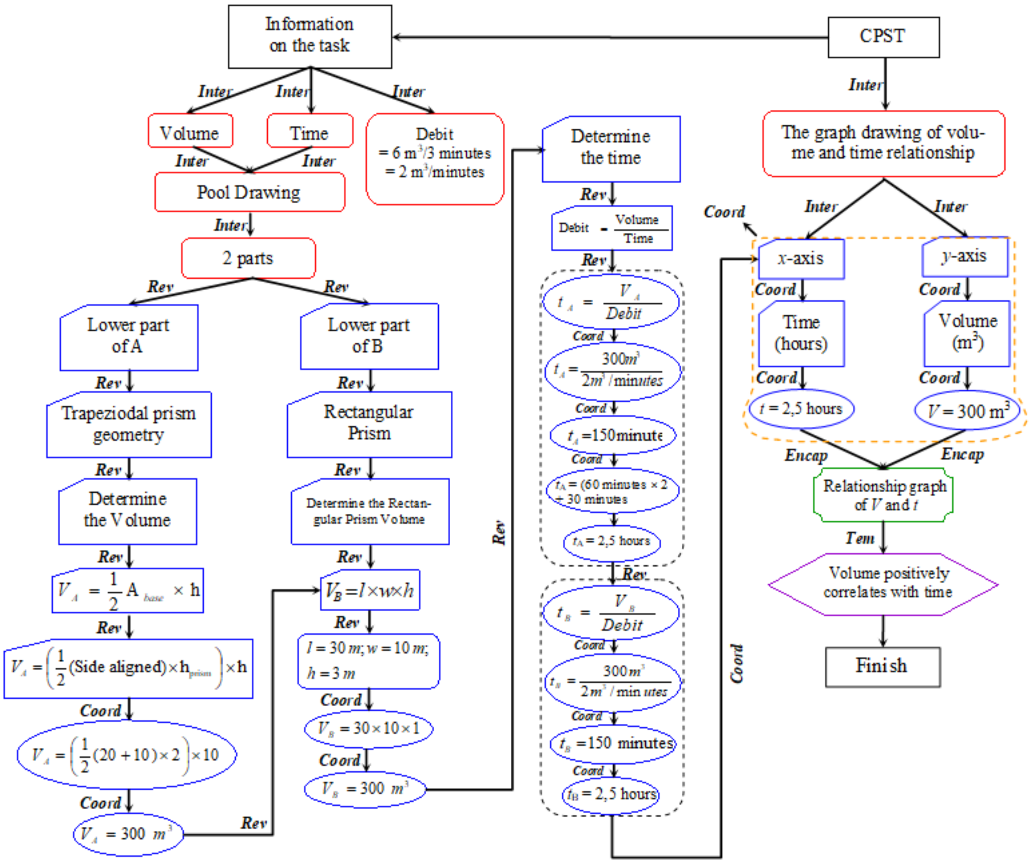


Figure 16. The Structure of Partial Quantitative Reasoning in Covariation Problem Solving according to APOS Theory

DISCUSSION

From the analysis that had been carried out on the data using APOS Theory, it was obtained that there were two categories of quantitative reasoning in covariation problem solving, namely holistic and partial covariation quantitative reasoning. The mental structure of the actions of the two categories showed the same thing. The subjects responded to the problem given by reading and understanding information, starting from the form of the problem, pictures, and commands that were in the task. This stage is an orientation and planning in understanding and defining the problema (Carlson & Bloom, 2005; Dwyer et al., 2003). At that stage, the subjects used a mental mechanism of interiorization to determine the quantities relationship. In planning problem solving, the subjects performed a mental mechanism of coordination, namely by partitioning the shape of the image in the task into several parts. Gradually, the quantities of the separate units were coordinated to get the values of the overall quantities (Ellis, 2011).

After the subject had done the planning by partitioning into several parts, then the subjects performed a mental reversal mechanism in which the subjects determined the mathematical method to solve the right problem in the process of problem solving. In this case, the subjects determined the formula for finding each volume of geometry. The problem solving process can use the arithmetic and algebraic approaches (Dwyer et al., 2003) and carry out quantitative reasoning inductively and deductively (Syarifuddin et al., 2019a).

Furthermore, in determining the volume quantity value, the subjects performed a mental coordination mechanism by substituting the quantity of each geometry into the volume formula, so as to obtain the volume quantity value of each geometry. In the process of quantitative reasoning, the subjects performed numerical quantitative operations (Thompson, 2011), namely by carrying out operations on quantities as a result of the additive combination of quantity networks, namely the sum of the quantities of geometry to obtain the overall quantities and the multiplicative combination of the internal relations of quantities, namely the quantities in a geometry consisting of quantities of length, width and height. At this stage, the subjects had not shown covariational reasoning of the problem because quantitative reasoning is the basis of covariational reasoning (Thompson, 2008).

In carrying out these quantitative operations, the subjects coordinated from different processes to find the volume of each geometry. Then from the coordination, the subjects continued by conducting a coordination mechanism to determine the volume of the fulfilled pool. When looking for this volume, there was coordination among the shapes of three geometries because they were in a single structural unit (pool). While the reversal process occurred when the subjects used the same formula in finding the volume of geometry. A process can be coordinated to other processes to build a new process and can also be reversed (Dubinsky, 1991; Figueroa et al., 2018; Moll et al., 2016), from the process of partitioning intact forms into several geometries, and then reversing the process through adding up the volume of each geometry to determine the overall volume.

After obtaining a fulfilled pool, the category subject in the holistic quantitative reasoning category encapsulated to determine the relationship between the volume quantity and the time quantity shown by drawing a Cartesian graph. In forming this relationship, the subject drew a Cartesian graph using only one volume value and one time value, so that the graph formed a straight line from the base point to the end point of the graph. This involved coordinating two quantities into a picture of continuous coordination as a continuous quantities, so that one could determine the value of quantity (Carlson et al., 2002). Students conceptualize a function as the process of mapping input values to output values (Carlson et al., 2015). This holistic covariation quantitative reasoning process is a global process of a function. In varying a quantity, one can simultaneously pay attention to the variety of changes intensity in quantity showing the relationship among covariation quantities (Johnson, 2012). To develop strong and global generalization about quantities relationship, students should make comparisons of quantities appearing related to the two initial quantities (Ellis, 2007). The role of quantities comparison is closely related to the ability of students to generalize correctly.

While the subject of partial quantitative reasoning category encapsulated to determine the relationship between volume quantity and time quantity by drawing a Cartesian graph using some volume quantity values and some time quantity values. These values were the pair of coordinate points of the parts that were partitioned. In drawing graphs, the subjects made intervals on each axis, in which the x-axis played role as time and the y-axis as volume. This describes that the operation of segmentation as the basis for generating the concept of measured quantities (Thompson, 2011), in this case students thought structurally related to numerical basis, involving composite units and multiplication relationships. The process of linking the quantities by the subjects of partial quantitative reasoning shows that the quantities of input used were referring to the partitioned parts before. In this case, the subjects connect quantity based on a discrete perspective (Clement, 1989; Johnson, 2012) which supports students in thinking about the function of composition (Carlson et al., 2015; Dubinsky & Harel, 1992). This quantitative reasoning ultimately shows the change of covariational reasoning process from one quantity to another quantity. Confrey & Smith (1995) characterizes covariational reasoning because it moves between the values of one successive variable, and then coordinates them by moving among the values of other consecutive variables. (Saldanha & Thompson, 1998) describe the picture of sequential and coordinated changes in two quantities as initial form of covariation developed into a picture of simultaneous change.

The reasoning shown by the subject in the above process is quantitative reasoning as the basis for giving birth to covariational reasoning. Quantitative reasoning plays an important role in covariational reasoning (Smith & Thompson, 2007; Thompson, 1990) because covariational reasoning is a cognitive activity involved in coordinating two varying quantities while paying attention to the way they change in relation to each other (Carlson et al., 2002; Confrey & Smith, 1995; Slavit, 1997).

The reasoning process is shown by the subject in doing this de-encapsulation as in the study of Carlson et al. (2002), in which students can represent image of the function of the dependent variable, which changes with the change in the independent variable. While Johnson (2012) describes that the reasoning of students who systematically vary a quantity can simultaneously pay attention to covariation in the intensity of changes in quantity, showing the relationship among quantities covariation. The reasoning of subjects in determining the quantity relation points is still due to the comparison of quantity values. This is in line with the results of Johnson (2015), in which students can make comparisons between the amount of changes in quantities.

CONCLUSION

The results of research on the quantitative reasoning process of students in covariation problem solving based on APOS Theory revealed that there were two quantitative reasoning process, namely holistic and partial covariation quantitative reasoning process. The holistic covariation quantitative reasoning process existed in the action phase, in which the students did the interiorization of reading and understanding the problem through paying full attention to the steps of linking quantities and planning problem solving. The next stage was the stage of the process, in which students determined mathematical methods to solve the right problem through mental coordination mechanisms, namely students partitioning the part of the whole to form quantities relationship. Then students did the mental reversal mechanism by recalling the formula corresponding to the previously partitioned form. Then the students returned to the mental reversal mechanism, which was to reuse the problem solving method that had been used in the previous coordination mechanism to be used in other partitions. In this case, students recalled the corresponding formula to be applied to the shape of geometry. To determine the relationship between quantities of volume and time, the students did a mental coordination mechanism, that was connecting the values of quantities by referring to the comparison of quantities as a whole, namely the students connected the quantities as a whole without looking at the partition that had been done before. As a result of this coordination, the subjects performed a mental mechanism of encapsulation by forming an image in the form of a continuous graph. The next encapsulation done by students was to interpret information by mentioning the quantities relationship formed from the graph, in which students explained that the graph was a straight-line graph and could be called a linear equation. At the schema stage, students applied mental mechanisms of thematization by understanding the concept of straight lines and the concept of linear equations.

The partial covariation quantitative reasoning process in the stages of action and process was the same as the stage carried out by students in the holistic covariation quantitative reasoning. After getting the values of the quantities of volume and time, then students continued with the mental mechanism of coordination connecting the values of the smaller parts of the quantities those were the partitions. In connecting the quantities, students did not pay attention to the comparison as a whole, but rather referred to the parts that had been partitioned. The results of the coordination were encapsulated to form a graphic image. The next encapsulation done by students was to interpret information by mentioning the quantities relationship formed from the graph, in which students explained that the graph was a straight-line graph and had a fixed comparison value. In the scheme stage, students carried out the mechanism of thematization by understanding the concept of straight lines and the concept of linear equations.

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